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No. 746

THE FREQUENCIES OF CANTILEVER WINGS IN BEAM AND TORSIONAL VIBRATIONS

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SUMMARY

Methods are described for calculating the period and frequency of vibration of cantilever wings and similar structures in which the weight and moment of inertia vary along the span. Both the beam and torsional frequencies may be calculated by these methods. The procedure is illustrated by examples.

It is shown that a surprisingly close approximation to the beam frequency may be obtained by a very brief calculation in which the curvature of the wing in vibration is assumed to be constant. A somewhat longer computation permits taking account of the true curvature of the beam by a series of successive approximations which are shown to be strongly convergent.

Analogous methods are applied to calculations of the torsional frequency. For the first approximation it is assumed that the angle of twist varies linearly along the semispan. The true variation of the twist is computed by successive approximations which are strongly convergent, as in the case of beam vibrations.

Notation

V, strain energy.

K, kinetic energy.

E, modulus of elasticity in tension.

G, modulus of elasticity in shear.

g, acceleration of gravity.

T, period of vibration.

N, frequency of vibration.

x, distance from root.

y, deflection.

R, radius of curvature.

v, velocity in beam vibration.

I, moment of inertia (beam).

w, weight per unit length.

P, polar moment of inertia of weight per unit length.

W, concentrated weight.

S, shearing force.

M, bending moment.

Q, torsional moment.

θ, angle of twist.

ω, angular velocity in torsional vibration.

L, length of cantilever.

A, area inclosed within cross-section of torque resistant structure.

s, perimeter of cross section.

t, thickness of shell.

Beam Vibration

At the instant of maximum amplitude of vibration, the beam has no motion and no kinetic energy. The strain energy in the bent beam is given by:

$$V = (1/2) \int EI(d^2y/dx^2)^2 dx \qquad (1)$$

The kinetic energy at zero strain energy is:

$$K = (1/2g) \int wv^2 dx \tag{2}$$

In simple harmonic motion:

$$v = 2\pi y/T \tag{3}$$

Substituting this value of v in (2) gives:

$$K = (2\pi^2/gT^2) \int wy^2 dx \tag{4}$$

$$V = K \tag{5}$$

whence:

$$T^2 = \frac{4\pi^2 \int wy^2 dx}{g \int EI(d^2y/dx^2)^2 dx} \tag{6}$$

$$N = 1/T \tag{7}$$

The distributed forces on the vibrating wing are equal to the mass times the acceleration. The acceleration is proportional to the deflection, so that the forces are proportional to the weight times the deflection. It is necessary only to determine the relative magnitudes of the deflections along the beam. Their absolute magnitudes do not influence the period.

The true form of the deflection curve can be computed quite accurately and readily by a series of successive approximations, beginning with a simple form of curve, such as one with a constant radius, i.e.:

$$d^2y/dx^2 = \text{constant}$$

In each successive approximation to the deflection curve, it is assumed that the distributed forces are proportional to the distributed weights multiplied by the deflections computed in the preceding approximation. The series of curves obtained in this manner converges very rapidly toward a final form. The calculations may be readily carried out by tabulations without actually drawing any curves. On the other hand, a graphical solution with the assistance of a mechanical integraph may be quicker than tabulations.

The procedure is shown by the following example.

Example 1.- Compute the period of vibration in the beam direction of a tapered cantilever wing in which the moments of inertia and the running weights at eleven equally spaced stations are as in table I. (Station numbers represent distances from the root.)

$$E = \overline{10}^7 \text{ lb./in.}^2$$

Method of Procedure

For the first approximation, the deflection y_1 in the last column of table I is assumed to be proportional to the square of the distance from the root.

The second approximation y_2 is calculated in table II by taking the running loads equal to wy_1. The shear and bending are then computed by successive integrations. Since relative and not actual values are required, and the station spacings are constant, the integrations consist merely of summations, as will be apparent from inspection of the table, without multiplying by the station intervals.

In accordance with beam theory:

$$d^2y/dx^2 = M/EI$$

The slope dy/dx is determined simply by summation of M/I, omitting division by the constant E and multiplication by the constant interval. The deflection y_2 is obtained by summation of dy/dx.

The summations for dy/dx and y are from the root outward (i.e., from the bottom upward in the table), while the summations for S and M are from the tip inward, or from the top downward in the table.

The variations in the ratio y_2/y_1 in the last column of table II show the differences between the first and second approximations to the form of the deflection curve.

The third approximation y_3 is determined in table III by the same procedure as in table II. The ratio y_3/y_2 is practically constant from tip to root, showing that the curve has settled down to its final form.

In table IV, d^2y/dx^2 is the M/I of table III. The calculation and summation of $I(d^2y/dx^2)^2$ from d^2y/dx^2 are obvious. The calculation and summation of wy^2 from y of table III are also obvious.

Substituting in equation (6) the numerical values of the summations in table IV:

$$T^2 = \frac{4\pi^2 \times 65517 \times 16 \times \overline{10}^7}{386 \times \overline{10}^7 \times 19160000}$$

$$= 0.005595 \text{ sec.}^2$$

$$T = 0.0748 \text{ sec.}$$

$$N = 13.36 \text{ v p s}$$

Approximate Solution

If it is assumed that $d^2y/dx^2 = 1$, corresponding to a constant radius of curvature, the calculation of T becomes the extremely simple and self-explanatory process shown in table V.

$$I(d^2y/dx^2)^2 = I$$

dy/dx divided by the station interval equals 1, 2, 3, 4, etc., from the bottom up, and y divided by the square of the interval is the summation of dy/dx from the bottom up.

$$T^2 = \frac{4\pi^2 \times 10067 \times 160000}{386 \times \overline{10}^7 \times 2965}$$

$$= 0.005556 \text{ sec.}^2$$

$$T = 0.0745 \text{ sec.}$$

$$N = 13.42 \text{ v p s}$$

The error in T and N is only 0.4 percent.

Example 2.- Since the very small error in the approximate method might be purely chance in this example, another calculation was made, using the same moments of inertia as in the previous example, but with an entirely different distribution of weight. The calculation of T by successive approximation is carried out in tables VI to X, inclusive, carrying the approximation to y one step further than before. From the summations in table X,

$$T^2 = \frac{4\pi^2 \times 18218 \times 16 \times \overline{10}^7}{386 \times \overline{10}^7 \times 1267700}$$

$$T^2 = 0.02352 \text{ sec.}^2$$

$$T = 0.1534 \text{ sec.}$$

$$N = 6.52 \text{ v p s}$$

Computation of T on the assumption that $d^2y/dx^2 = 1$ is carried out in table XI.

$$T^2 = \frac{4\pi^2 \times 42576 \times 160000}{386 \times \overline{10}^7 \times 2965}$$

$$= 0.02350 \text{ sec.}^2$$

$$T = 0.1533 \text{ sec.}$$

$$N = 6.52 \text{ v p s}$$

The error by the approximate solution is even less than in the preceding example.

From these two examples, it may be concluded that the approximate method is satisfactory for most cantilever wings. It may be noted from tables IV and X that in the first example the true values of d^2y/dx^2 vary about 80 percent between the root and the tip, while in the second example the variation is only 50 percent, which accounts for the closer results in the second example than in the first; but it is still rather remarkable that the approximate method should be so accurate for curvature that varies from 50 percent to 80 percent from the assumed constant value.

Example 3.- Compute the period of vibration of a weightless cantilever of uniform $I = 300$ in.^{4}, carrying a concentrated load of 1,000 pounds at 200 inches from the root. $E = \overline{10}^7$ lb./in.^{2}.

The exact solution is:

$$T = \sqrt{y}/3.13$$

where y is the static deflection at the weight.

$$y = \frac{WL^3}{EI} = \frac{1000 \times \overline{200}^3}{900 \times \overline{10}^7} = 0.89 \text{ in.}$$

$$T = \sqrt{0.89}/3.13 = 0.302 \text{ sec.}$$

Assuming $d^2y/dx^2 = 1/R = \text{const.}$

At tip, $y = L^2/2R = 20000/R$

$$\int wy^2dx = 1000 \times (20000/R)^2$$

$$\int EI(d^2y/dx^2)^2 \, dx = \overline{10}^7 \times 300 \times 200/R^2$$

$$T^2 = \frac{4\pi^2 \times 1000 \times (20000/R)^2}{386 \times \overline{10}^7 \times 300 \times 200/R^2}$$

$$= 0.068 \text{ sec.}^2$$

$$T = 0.261 \text{ sec.}$$

In this case the error is large, as might be expected from the fact that the actual d^2y/dx^2 varies as x instead of being constant. In other words, for a good approximation to the period or frequency, the assumed curve of deflection must not be too far from reality, although as shown in the examples it can be quite surprisingly far without appreciable error in the result.

Torsional Frequency

The twist of a thin-walled closed section under torsional moments is given by:

$$\frac{d\theta}{dx} = \frac{Q}{4A^2G}\int \frac{ds}{t} \tag{8}$$

The strain energy of the twisted sections is given by:

$$\frac{dV}{dx} = \frac{Q}{2}\frac{d\theta}{dx} \tag{9}$$

Let

$$k = (1/4A^2) \int (1/t)\, ds \tag{10}$$

Combining (8), (9), and (10),

$$V = \frac{G}{2}\int \frac{1}{k}\left(\frac{d\theta}{dx}\right)^2 dx \tag{11}$$

The kinetic energy is given by:

$$\frac{dK}{dx} = \frac{P\omega^2}{2g} = \frac{2\pi^2 P\theta^2}{gT^2}$$

$$K = \frac{2\pi^2}{gT^2}\int P\theta^2 dx \tag{12}$$

$$V = K$$

whence

$$T^2 = \frac{4\pi^2 \int P\theta^2 dx}{gG \int (1/k)\cdot(d\theta/dx)^2 dx} \tag{13}$$

The period may be calculated by a procedure analogous to that used for beam vibrations, beginning with the assumption that $d\theta/dx = 1$, and finding the true form of the twist by successive approximations.

Example 4.- Find the torsional period of vibration of the cantilever wing having the characteristics given in table XII.

Let $G = 4{,}000{,}000$ lb./in.^{2}

In the first approximation, θ is assumed proportional to the distance from the root.

In the second approximation, the applied torsional moments are taken equal to $P\theta$ at each station. The total torque Q_2 is obtained by summation of these moments (table XIII). The twist is given by $d\theta/dx = kQ/G$.

Summation of kQ_2 from the root outward gives θ_2, omitting division by G.

It is seen from the last column of table XIII that θ_2/θ_1 is a reasonably constant ratio; but the successive approximations are carried one step farther in table XIV. The variation in θ_3/θ_2 is negligible.

The summations of $(1/k)(d\theta/dx)^2$ and $P\theta^2$ are calculated in table XV, using the values of $d\theta/dx$ and θ from table XIV.

The numerical values from table XV are substituted in equation (13), giving the torsional period:

$$T^2 = \frac{4\pi^2 \times 132464 \times 4 \times \overline{10}^5}{386 \times 4 \times \overline{10}^6 \times 6559 \times \overline{10}^3}$$

$$= 0.000206 \text{ sec.}^2$$

$$T = 0.01435 \text{ sec.}$$

$$N = 69.6 \text{ v p s}$$

The short-cut or approximate calculation based on the assumption that $d\theta/dx = 1$ is carried out in table XVI. Only the figures in the last column and the summation of the second column require any computation. All the others may be written down directly. Substituting the numerical values in equation (13),

$$T^2 = \frac{4\pi^2 \times 82120 \times 400}{386 \times 4 \times \overline{10}^6 \times 4157}$$

$$= 0.000202 \text{ sec.}^2$$

$$T = 0.0142 \text{ sec.}$$

$$N = 70.4 \text{ v p s}$$

The error is only 1.0 percent.

Bureau of Aeronautics, Navy Department,
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TABLE I

Station	I	w	y_1
220	2	0.53	484
200	6	0.72	400
180	22	0.91	324
160	52	1.10	256
140	96	1.29	196
120	156	1.48	144
100	254	1.67	100
80	329	1.86	64
60	428	2.05	36
40	480	2.24	16
20	546	2.45	4
0	620	2.62	0

TABLE II

Station	wy_1	S_2	M_2	M_2/I	dy/dx	y_2	y_2/y_1
220	256	256	256	42.9	307.8	1664.1	3.44
200	288	544	800	36.4	264.7	1356.3	3.39
180	295	839	1639	31.5	228.3	1091.6	3.37
160	282	1121	2760	28.8	196.8	863.5	3.38
140	253	1374	4134	26.5	168.0	666.7	3.40
120	213	1587	5721	24.4	141.5	498.7	3.46
100	167	1754	7475	22.7	117.1	357.2	3.57
80	119	1873	9348	22.1	94.4	240.1	3.75
60	74	1947	11295	23.5	72.3	145.7	4.05
40	36	1983	13278	24.2	48.8	73.4	4.09
20	10	1993	15271	24.6	24.6	24.6	6.15
0	0				1.3	0	-

TABLE III

Station	wy_2	S_3	M_3	M_3/I	dy/dx	y_3	y_3/y_2
220	882	882	882	147	1017.2	5467	3.30
200	875	1757	2639	120	870.2	4470	3.30
180	995	2752	5391	103.6	750.2	3600	3.30
160	950	3602 3702	8993	93.7	646.6	2850	3.30
140	860	4462	13455	86.2	552.9	2203	3.30
120	738	5200	18655	79.7	466.7	1650	3.31
100	596	5796	24451	74.3	387.0	1184	3.32
80	447	6243	30694	72.8	312.7	796.7	3.32
60	298	6541	37235	77.6	239.9	484.0	3.32
40	164	6705	43940	80.5	162.3	244.1	3.32
20	60	6765	50705	81.6	81.8	81.8	3.32
0	0					0	-

TABLE IV

Station	$\frac{d^2y}{dx^2}$	$\left(\frac{d^2y}{dx^2}\right)^2$	$\frac{I\left(\frac{d^2y}{dx^2}\right)^2}{1000}$	$\frac{y}{400}$	$\frac{y^2}{16\times10^7}$	$\frac{wy^2}{16\times10^7}$
220	147	21600	130	5467	30,100	15950
200	120	14400	317	4470	20,000	14400
180	103.6	10720	558	3600	12,960	11800
160	93.7	8790	844	2850	8,120	8940
140	86.2	7440	1161	2203	4,850	6250
120	79.7	6360	1490	1650	2,720	4020
100	74.3	5520	1820	1184	1,405	2340
80	72.8	5300	2240	798	637	1185
60	77.6	6030	2900	484	234	480
40	80.5	6490	3540	244	60	135
20	81.6	6700	4160	82	7	17
0				0	0	0
			19,160			65,517

TABLE V

Station	$I\left(\frac{d^2y}{dx^2}\right)^2$	$\frac{dy/dx}{20}$	$\frac{y}{400}$	$\frac{y^2}{160,000}$	$\frac{wy^2}{160,000}$
220	2		66	4336	2298
		11			
200	6		55	3025	2178
		10			
180	22		45	2025	1843
		9			
160	52		36	1296	1426
		8			
140	96		28	784	1011
		7			
120	156		21	441	653
		6			
100	234		15	225	376
		5			
80	329		10	100	186
		4			
60	422		6	36	74
		3			
40	480		3	9	20
		2			
20	546		1	1	2
		1			
0	620		0	0	0
	2965				10,067

TABLE VI

Station	I in.^{4}	w lb/in.	y_1 in.
220	2	2.0	484
200	6	2.0	400
180	22	5.0	324
160	52	4.0	256
140	96	8.0	196
120	156	7.0	144
100	234	7.0	100
80	329	10.0	64
60	422	8.0	36
40	480	12.0	16
20	546	15.0	4
0	620	20.0	0

TABLE VII

Station	wy_1	S_2	M_2	M_2/I	dy/dx	y_2	y_2/y_1
220	968					666.8	1.38
		97			118.1		
200	800		97	16.2		548.7	1.37
		177			101.9		
180	1620		274	12.5		446.8	1.38
		340			89.4		
160	1024		614	11.8		357.4	1.40
		441			77.6		
140	1568		1055	11.0		279.8	1.43
		598			66.6		
120	1008		1653	10.6		213.2	1.48
		699			60.0		
100	700		2352	10.1		153.2	1.53
		769			49.9		
80	640		3121	9.5		103.3	1.61
		833			40.4		
60	288		3954	9.3		62.9	1.75
		862			31.1		
40	192		4816	10.0		31.8	1.98
		881			21.1		
20	60		5697	10.4		10.7	2.67
		941			10.7		
0	0		6638	10.7		0	–

TABLE VIII

Station	wy_2	S_3	M_3	M_3/I	dy/dx	y_3	y_3/y_2
220	1334					963.2	1.445
		133			169.9		
200	1097		133	22.2		793.6	1.446
		243			147.7		
180	2234		376	17.1		645.9	1.446
		467			130.6		
160	1430		843	16.2		515.3	1.442
		609			114.4		
140	2238		1452	15.1		400.9	1.433
		833			99.3		
120	1492		2285	14.7		301.6	1.415
		983			84.6		
100	1073		3268	14.0		217.0	1.416
		1099			70.6		
80	1033		4367	13.3		146.4	1.417
		1193			57.3		
60	503		5560	13.2		89.1	1.417
		1243			44.1		
40	582		6803	14.2		45.0	1.415
		1282			29.9		
20	161		8085	14.8		15.1	1.411
		1298			15.1		
0	0		9383	15.1		0	–

TABLE IX

Station	wy_3	S_4	M_4	M_4/I	dy/dx	y_4	y_4/y_3
220	1926					1385.6	1.439
		193			245.2		
200	1587		193	32.6		1140.6	1.437
		351			212.6		
180	3230		544	24.7		928.0	1.437
		674			187.9		
160	2060		1218	23.4		740.1	1.436
		880			164.5		
140	3807		2098	21.8		576.6	1.436
		1201			142.7		
120	2111		3299	21.2		432.9	1.435
		1412			121.5		
100	1519		4711	20.2		311.4	1.435
		1564			101.5		
80	1464		6275	19.1		210.1	1.435
		1710			82.2		
60	713		7985	18.9		127.9	1.435
		1782			63.3		
40	540		9767	20.4		64.6	1.439
		1856			42.9		
20	226		11603	21.3		21.7	1.437
		1858			21.7		
0	0		13461	21.7		0	-

TABLE X.

Station	$\frac{d^2y}{dx^2}$	$\left(\frac{d^2y}{dx^2}\right)^2$	$I\left(\frac{d^2y}{dx^2}\right)^2$	$\frac{y}{400}$	$\frac{y^2}{16x10^7}$	$\frac{wy^2}{16x10^7}$
220				1386	1921	3842
200	32.8	1062	6,372	1141	1302	2604
180	24.7	610	13,420	928	861	4305
160	23.4	548	28,496	740	548	2192
140	21.8	475	45,600	576	338	2666
120	21.2	450	70,200	455	187.5	1312
100	20.2	408	95,472	311	96.7	677
80	19.1	365	120,065	210	44.1	441
60	18.9	357	150,654	128	16.4	131
40	20.4	416	199,680	65	4.2	50
20	21.2	450	245,700	22	0.5	8
0	21.7	471	292,020	0	0	0
			1,267,699			18,218

TABLE XI.

Station	$I\left(\frac{d^2y}{dx^2}\right)^2$	$\frac{dy}{dx} \div 20$	$\frac{y}{400}$	$\frac{y^2}{160,000}$	$\frac{wy^2}{160,000}$
220	2		66	4356	8672
		11			
200	6		55	3025	6050
		10			
180	22		45	2025	10125
		9			
160	52		36	1296	5184
		8			
140	96		28	784	6272
		7			
120	156		21	441	3287
		6			
100	254		15	225	1575
		5			
80	389		10	100	1000
		4			
60	428		6	36	288
		3			
40	480		3	9	108
		2			
20	548		1	1	15
		1			
0	620		0	0	0
	2965				42,576

TABLE XII

Station	A	$\int\frac{ds}{t}$	$\frac{1}{k}$	P	θ_1
220	25	1000	2.5	13	11
200	70	1320	14.8	52	10
180	130	1740	38.8	115	9
160	195	1625	93.6	186	8
140	255	1825	143	272	7
120	320	1755	233	356	6
100	380	1860	310	440	5
80	440	1950	401	522	4
60	500	1725	580	593	3
40	550	1725	700	648	2
20	590	1725	806	703	1
0	600	1725	836	750	0

Tables 9,10,11,12

TABLE XIII

Station	$P\theta_1$	Q_2	$d\theta/dx = kQ_2$	θ_2	θ_2/θ_1
220	143			211.1	19.2
200	520	143	9.7	201.4	20.1
180	1035	663	17.1	184.3	20.5
160	1488	1698	18.1	166.2	20.8
140	1904	3186	22.3	143.9	20.6
120	2136	5090	21.8	122.1	20.4
100	2200	7226	23.3	98.8	19.8
80	2088	9426	23.5	75.3	18.9
60	1779	11514	19.9	55.4	18.5
40	1296	13293	19.0	36.4	18.2
20	703	14589	18.1	18.3	18.3
0	0	15292	18.3	0	–

TABLE XIV

Station	$P\theta_2$	Q_3	$d\theta/dx = kQ_3$	θ_3	θ_3/θ_2
220	274			424.2	2.01
200	1047	274	16.5	405.7	2.02
180	2119	1321	34.0	371.7	2.02
160	3091	3440	36.8	334.9	2.02
140	3914	6531	45.7	289.2	2.01
120	4347	10445	44.9	244.3	2.00
100	4347	14792	47.7	196.6	1.99
80	3931	19139	47.7	148.9	1.98
60	3285	23070	39.8	109.1	1.97
40	2359	26355	37.6	71.5	1.96
20	1286	28714	36.6	35.9	1.96
0	0	30000	35.9	0	–

TABLE XV

Station	$\frac{d\theta}{dx}$	$\left(\frac{d\theta}{dx}\right)^2$	$\frac{\frac{1}{k}\left(\frac{d\theta}{dx}\right)^2}{1000}$	$\frac{\theta}{20}$	$\frac{\theta}{400,000}$	$\frac{P\theta^2}{400,000}$
220				424.2	179.9	2339
200	18.5	342	5	405.7	164.6	8559
180	34.0	1156	45	371.7	138.2	15893
160	36.8	1354	127	334.9	112.2	20869
140	45.7	2088	299	289.2	83.6	22739
120	44.9	2016	470	244.3	59.7	21253
100	47.7	2275	705	196.6	38.7	17028
80	47.7	2275	912	148.9	22.2	11588
60	39.8	1584	919	109.1	11.9	7711
40	37.6	1414	990	71.5	5.1	3565
20	35.6	1267	1021	35.9	1.2	900
0	35.9	1289	1076	0	0	0
			6569			132,464

TABLE XVI

Station	$\frac{1}{k}\left(\frac{d\theta}{dx}\right)^2$	$\frac{\theta}{20}$	$\frac{\theta^2}{400}$	$\frac{P\theta^2}{400}$
220	2.5	11	121	1573
200	14.8	10	100	5200
180	38.8	9	81	9315
160	93.6	8	64	11904
140	143	7	49	13328
120	233	6	36	12816
100	310	5	25	11000
80	401	4	16	8352
60	560	3	9	5337
40	700	2	4	2592
20	806	1	1	703
0	835	0	0	0
	4157			82120